Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number

Candidate Number

Pearson Edexcel Level 3 GCE

Thursday 22 June 2023

Afternoon (Time: 1 hour 30 minutes)

Paper reference **9FM0/3A**

Further Mathematics

Advanced

PAPER 3A: Further Pure Mathematics 1

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need*.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 7 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

P 7 2 7 9 6 R A 0 1 2 8

Pearson

1. (a) Use Simpson's rule with 4 intervals to find an estimate for

$$\int_0^2 e^{\sin^2 x}\,dx$$

Give your answer to 3 significant figures.

(4)

Given that $\int_0^2 e^{\sin^2 x}\,dx = 3.855$ to 4 significant figures,

(b) comment on the accuracy of your answer to part (a).

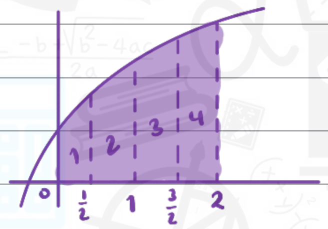
(1)

only used as an estimate

a. Simpson's Rule : $\int_a^b f(x)\,dx \approx \frac{h}{3}\left[y_0 + y_n + 2(y_2 + y_4 + \dots + y_{n-2}) + 4(y_1 + y_3 + \dots + y_{n-1})\right]$

where $h = \frac{b-a}{n}$

$h = \frac{(2)-(0)}{4} = \frac{2}{4} = 0.5$



* note that Simpson's rule is not given in formulae booklet. - must memorise.

Area : $\frac{1}{3}h\left(\text{ends} + 4(\text{odd}) + 2(\text{even})\right)$

Plot a table of values for x and y.

x	0	0.5	1.0	1.5	2.0
y	1	$e^{\sin^2(0.5)}$	$e^{\sin^2(1.0)}$	$e^{\sin^2(1.5)}$	$e^{\sin^2(2.0)}$

use exact values and only round at the end.

always in radians

Area : $\frac{1}{3}(0.5)\left(1 + e^{\sin^2(2.0)} + 4\left(e^{\sin^2(0.5)} + e^{\sin^2(1.5)}\right) + 2\left(e^{\sin^2(1.0)}\right)\right)$

Area ≈ 3.87 units^2 (3 s.f.)

b. Accurate to 2 significant figures.

9.2: 2nd Order DE's   9.3: Modelling with DE's

2. The vertical height, h m, above horizontal ground, of a passenger on a fairground ride, t seconds after the ride starts, where $t \leqslant 5$, is modelled by the differential equation

$$t^2\frac{d^2h}{dt^2} - 2t\frac{dh}{dt} + 2h = t^3 \qquad \text{(I)}$$

(a) Given that $t = e^x$, show that

(i) $t\dfrac{dh}{dt} = \dfrac{dh}{dx}$

(ii) $t^2\dfrac{d^2h}{dt^2} = \dfrac{d^2h}{dx^2} - \dfrac{dh}{dx}$

(4)

(b) Hence show that the transformation $t = e^x$ transforms equation (I) into the equation

$$\frac{d^2h}{dx^2} - 3\frac{dh}{dx} + 2h = e^{3x}$$

(1)

(c) Hence show that

$$h = At + Bt^2 + \frac{1}{2}t^3$$

where A and B are constants.

(6)

Given that when $t = 1$, $h = 2.5$ and when $t = 2$, $\dfrac{dh}{dt} = -1$

(d) determine the height of the passenger above the ground 5 seconds after the start of the ride.

(5)

a.

$\frac{dh}{dx} = \frac{dh}{dt} \times \frac{dt}{dx}$ another way $\frac{dh}{dx}$ can be expressed.   notice that both dt terms will cancel out.

$t = e^x$ differentiate both sides w.r.t. x.

$\frac{dt}{dx} = e^x = t$

sub back into new expression of $\frac{dh}{dx}$ we made.

$\therefore t\frac{dh}{dt} = \frac{dh}{dx}$ // (shown)

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Question 2 continued

b. Take new expression $t\frac{dh}{dt} = \frac{dh}{dx}$, and differentiate both sides with respect to t.

$t\frac{dh}{dt} = \frac{dh}{dx}$

$\frac{d}{dt}\left(\frac{dh}{dx}\right) = \frac{d^2h}{dx^2}\left(\frac{dx}{dt}\right)$

$\frac{dh}{dt} + t\frac{d^2h}{dt^2} = \left(\frac{d^2h}{dx^2}\right)\left(\frac{dx}{dt}\right)$

$\downarrow$ same as $\frac{1}{t}$ from ai.

$\frac{dh}{dt} + t\frac{d^2h}{dt^2} = \frac{d^2h}{dx^2}\left(\frac{1}{t}\right)$ $(\times t)$

$t^2\frac{d^2h}{dt^2} + t\frac{dh}{dt} = \frac{d^2h}{dx^2}$

$\downarrow$ same as $\frac{dh}{dx}$ from (ai.)

$t^2\frac{d^2h}{dt^2} + \frac{dh}{dx} = \frac{d^2h}{dx^2}$

$t^2\frac{d^2h}{dt^2} = \frac{d^2h}{dx^2} - \frac{dh}{dx}$ // (shown)

b. $t^2\frac{d^2h}{dt^2} - 2t\frac{dh}{dt} + 2h = t^3$

$\left(\frac{d^2h}{dx^2} - \frac{dh}{dx}\right) - \left(2\frac{dh}{dx}\right) + 2h = (e^x)^3$

$\frac{d^2h}{dx^2} - 3\frac{dh}{dx} + 2h = e^{3x}$ // (shown)

c. solve auxillary eqⁿ.

$m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0$

$m = 1$, $m = 2$

Question 2 continued

$h = Ae^{x} + Be^{2x}$ (Complementary function C.F.)

The 2^{nd} order D.E is non-homogenous (on RHS e^{3x} instead of 0)

↳ Must find particular integral P.I

P.I: let $h = ke^{3x}$

$h' = 3ke^{3x}$

$h'' = 9ke^{3x}$

$(9ke^{3x}) - 3(3ke^{3x}) + 2(ke^{3x}) = e^{3x}$

$9ke^{3x} - 9ke^{3x} + 2ke^{3x} = e^{3x}$

$2ke^{3x} = e^{3x}$

$2k = 1$

$k = 1/2$

General sol^n: C.F. + P.I.

gen sol^n: $h = Ae^{x} + Be^{2x} + \frac{1}{2}e^{3x}$

$t = e^{x}$ so convert eq^n in terms of h and t.

$h = A(e^{x}) + B(e^{x})^2 + \frac{1}{2}(e^{x})^3$

$h = At + Bt^2 + \frac{1}{2}t^3$ // (shown)

a. ① $t = 1$, $h = 2.5$ || ② $t = 2$ $\frac{dh}{dt} = -1$ (given in question)

① $2.5 = A(1) + B(1) + \frac{1}{2}(1)^3$

$2.5 = A + B + 0.5$

$A + B = 2$

② $\frac{dh}{dt} = A + 2Bt + \frac{3}{2}t^2$

$-1 = A + 2B(2) + \frac{3}{2}(2)^2$

$-1 = A + 4B + 6$

$-7 = A + 4B$

solve these 2 eq^{n}s simultaneously

Question 2 continued

$A + B = 2$

$A + 4B = -7$ ⊖

$3B = -9$

$B = -3$

$A + (-3) = 2$

$A - 3 = 2$

$A = 5$

$h = 5t - 3t^2 + \frac{1}{2}t^3$

@ $t = 5$

$5(5) - 3(5)^2 + \frac{1}{2}(5)^3$

$= 12.5m$

12.5m

(Total for Question 2 is 16 marks)

4.2: Using Graphs To Solve Inequalities | 4.3: Modulus Inequalities

3. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

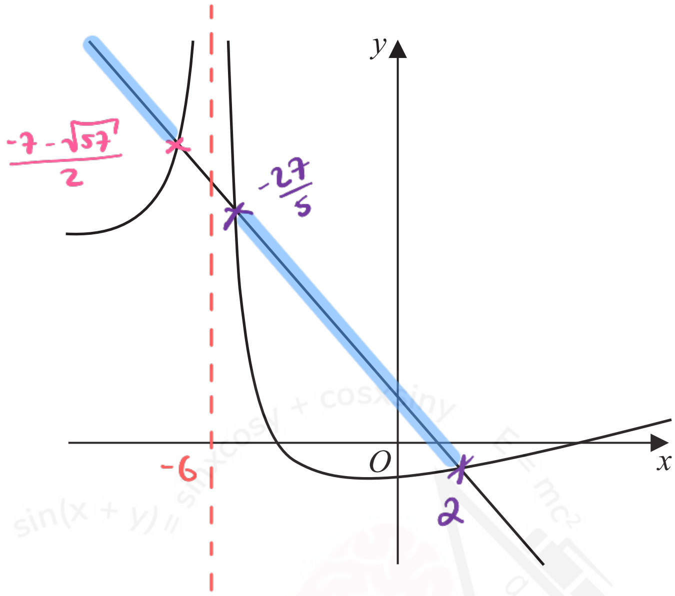

Figure 1

Figure 1 shows a sketch of the curve with equation $y = \dfrac{x^2 - 2x - 24}{|x+6|}$ and the line with equation $y = 5 - 4x$

Use algebra to determine the values of x for which

$$\frac{x^2 - 2x - 24}{|x+6|} < 5 - 4x$$

(7)

→ highlight area on graph which inequality satisfies

For modulus, 2 scenarios we need to solve for:

(1)

$$\frac{x^2 - 2x - 24}{x+6} < 5 - 4x$$

(2)

$$\frac{x^2 - 2x - 24}{-(x+6)} < 5 - 4x$$

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Question 3 continued

(1) $\frac{x^2-2x-24}{x+6} < 5-4x$ will give sol^{n}s for $x > -6$

$\frac{x^2-2x-24}{\cancel{x+6}} \times (x+6)^{\cancel{2}} < 5-4x \times (x+6)^2$

$(x-6)(x+4)(x+6) < (5-4x)(x+6)^2$

$(x-6)(x+4)(x+6) - (5-4x)(x+6)^2 < 0$

$(x+6)\left[(x-6)(x+4) - (5-4x)(x+6)\right] < 0$

$(x+6)\left[(x^2-2x-24) - (-4x^2-19x+30)\right] < 0$

$(x+6)(5x^2+17x-54) < 0$

$(x+6)(5x+27)(x-2) < 0$

$x = -6 \quad x = -27/5 \quad x = 2$

@ $x = -6$, asymptote

L If you sub $x = -6$ into $y = \frac{x^2-2x-24}{|x+6|}$, y is undefined.

(2) $\frac{x^2-2x-24}{-x-6} < 5-4x$ will give sol^{n}s for $x < -6$

$\frac{x^2-2x-24}{\cancel{(-x-6)}} \times (-x-6)^{\cancel{2}} < (5-4x)(-x-6)^2$

$(x-6)(x+4)(-x-6) < (5-4x)(-x-6)^2$

$(x-6)(x+4)(-x-6) - (5-4x)(-x-6)^2 < 0$

$(-x-6)\left[(x-6)(x+4) - (5-4x)(-x-6)\right] < 0$

Question 3 continued

$(-x-6)\left[(x^2-2x-24)-(4x^2+19x-30)\right]<0$

$(-x-6)(-3x^2-21x+6)<0$

$\downarrow$ $x=-6$

$\downarrow$ Solve using quadratic formulae.

$$x = \frac{-(-21) \pm \sqrt{(-21)^2-(4)(-3)(6)}}{(2)(-3)}$$

$$x = \frac{-7 \pm \sqrt{57}}{2}$$

$$\frac{-7-\sqrt{57}}{2} \approx -7.27$$

$\uparrow$

$\therefore$ This is intersection point on left.

$x < \frac{-7-\sqrt{57}}{2}$, $-\frac{27}{5} < x < 2$

3.1: Ellipses
3.3: Eccentricity
3.4: Tangents & Normals To An Ellipse
3.6: Loci

4. The ellipse E has equation

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

(a) Determine the exact value of the eccentricity of E

(2)

The points $P(4\cos\theta, 3\sin\theta)$ and $Q(4\cos\theta, -3\sin\theta)$ lie on E where $0 < \theta < \frac{\pi}{2}$
The line l_1 is the normal to E at the point P

(b) Use calculus to show that l_1 has equation

$$4x\sin\theta - 3y\cos\theta = 7\sin\theta\cos\theta$$

(4)

The line l_2 passes through the origin and the point Q
The lines l_1 and l_2 intersect at the point R

(c) Determine, in simplest form, the coordinates of R

(4)

(d) Hence show that, as θ varies, R lies on an ellipse which has the same eccentricity as ellipse E

(2)

a. $\frac{x^2}{(4)^2} + \frac{y^2}{(3)^2} = 1$

$a = 4 \quad b = 3$

eccentricity formulae (given in F.B)

"$b^2 = a^2(1-e^2)$" $\quad 0 < e < 1$

this part not included in F.B. but is true.

$9 = 16(1-e^2)$

$\frac{9}{16} = 1 - e^2$

$e^2 = \frac{7}{16}$

$e = \pm\frac{\sqrt{7}}{4}$

$e \neq -\frac{\sqrt{7}}{4}$

$\therefore e = \frac{\sqrt{7}}{4}$

taken from F.B.

Conics

	Ellipse	Parabola	Hyperbola	Rectangular Hyperbola
Standard Form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$y^2 = 4ax$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$xy = c^2$
Parametric Form	$(a\cos\theta, b\sin\theta)$	$(at^2, 2at)$	$(a\sec\theta, b\tan\theta)$ $(\pm a\cosh\theta, b\sinh\theta)$	$\left(ct, \frac{c}{t}\right)$
Eccentricity	$e < 1$ $b^2 = a^2(1-e^2)$	$e = 1$	$e > 1$ $b^2 = a^2(e^2 - 1)$	$e = \sqrt{2}$
Foci	$(\pm ae, 0)$	$(a, 0)$	$(\pm ae, 0)$	$(\pm\sqrt{2}c, \pm\sqrt{2}c)$
Directrices	$x = \pm\frac{a}{e}$	$x = -a$	$x = \pm\frac{a}{e}$	$x + y = \pm\sqrt{2}c$
Asymptotes	none	none	$\frac{x}{a} = \pm\frac{y}{b}$	$x = 0, y = 0$

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Question 4 continued

b. differentiate ellipse eqⁿ.

$$\frac{2x}{16} + \frac{2y}{9}\left(\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx}\left(\frac{2y}{9}\right) = -\frac{x}{8}$$

$\div \frac{2y}{9}$ on both sides

$$\frac{dy}{dx} = -\frac{x}{8} \times \frac{9}{2y}$$

$$\frac{dy}{dx} = -\frac{9x}{16y}$$

$$\frac{dy}{dx} = -\frac{\cancel{9}^{3}(4\cos\theta)}{\cancel{16}_{4}(\cancel{3}\sin\theta)} = \frac{-3\cos\theta}{4\sin\theta}$$

$M_{tangent}$

$$M_{normal} = \frac{4\sin\theta}{3\cos\theta}$$

$$y - (3\sin\theta) = \frac{4\sin\theta}{3\cos\theta}(x - 4\cos\theta)$$

$\times(3\cos\theta)$

rearrange to get wanted eqⁿ.

$$3y\cos\theta - 9\sin\theta\cos\theta = 4x\sin\theta - 16\sin\theta\cos\theta$$

$$4x\sin\theta - 3y\cos\theta = 7\sin\theta\cos\theta \text{ (shown)}$$

c. Origin $(0,0)$ Q: $(4\cos\theta, -3\sin\theta)$

$$M_{L_2} = \frac{-3\sin\theta - 0}{4\cos\theta - 0} = \frac{-3\sin\theta}{4\cos\theta}$$

$$y - (0) = -\frac{3\sin\theta}{4\cos\theta}(x - 0)$$

$$L_2: \quad y = \frac{-3\sin\theta}{4\cos\theta}x$$

Sub y into L_1 eqⁿ and solve for x.

$$4x\sin\theta - 3\left(\frac{-3\sin\theta}{4\cancel{\cos\theta}}x\right)\cancel{\cos\theta} = 7\sin\theta\cos\theta$$

$$4x\sin\theta + \frac{9}{4}x\sin\theta = 7\sin\theta\cos\theta$$

$$\frac{25}{4}x\cancel{\sin\theta} = 7\cancel{\sin\theta}\cos\theta$$

(cancel out sinθ terms on both sides)

Question 4 continued

$\frac{25}{4}x = 7\cos\theta$

$x = \frac{28}{25}\cos\theta$

(sub into L_2 (easier) and solve for y.

$y = \frac{-3\sin\theta}{\cancel{4}\cancel{\cos\theta}} \times \frac{\cancel{28}^{7}\cancel{\cos\theta}}{25}$

$y = -\frac{21}{25}\sin\theta$

$R = \left(\frac{28}{25}\cos\theta, -\frac{21}{25}\sin\theta\right)$

a. If R lies on ellipse, create new ellipse D.

ellipse D eq^n: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

F.B.: $(a\cos\theta, b\sin\theta) \equiv \left(\frac{28}{25}\cos\theta, -\frac{21}{25}\sin\theta\right)$

$a = \frac{28}{25}$, $b = \frac{-21}{25}$

$b^2 = a^2(1-e^2)$

$\left(-\frac{21}{25}\right)^2 = \left(\frac{28}{25}\right)^2(1-e^2)$

$\frac{\left(-\frac{21}{25}\right)^2}{\left(\frac{28}{25}\right)^2} = (1-e^2)$

$(1-e^2) = \frac{9}{16}$

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Question 4 continued

$\frac{7}{16} = e^2$

$e = \pm\frac{\sqrt{7}}{4}$

$0 < e < 1$

$\therefore e = \frac{\sqrt{7}}{4}$ // (shown)

(Total for Question 4 is 12 marks)

P 7 2 7 9 6 R A 0 1 5 2 8

5. (a) Show that the substitution $t = \tan\left(\frac{x}{2}\right)$ transforms the integral

$$\int \frac{1}{2\sin x - \cos x + 5}\,dx$$

into the integral

$$\int \frac{1}{3t^2 + 2t + 2}\,dt$$

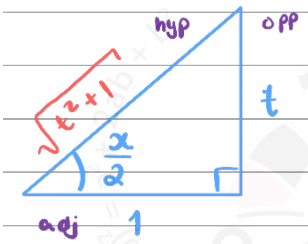
(4)

(b) Hence determine

$$\int \frac{1}{2\sin x - \cos x + 5}\,dx$$

(4)

a. deriving sin(x), cos(x), and tan(x) from $t = \tan\left(\frac{x}{2}\right)$



1) draw a right-angled triangle and label angle and sides which you know.

* you are allowed to memorise the t-formulae for sin(x), cos(x), tan(x) and do not have to derive it in the exam unless specifically asked.

2) work out hypotenuse in terms of t, (pythagoras)

$\sqrt{(t)^2+(1)^2}$

$= \sqrt{t^2+1}$

3) label each side of triangle opposite, adjacent, hypotenuse

4) write out $\sin\left(\frac{x}{2}\right), \cos\left(\frac{x}{2}\right), \tan\left(\frac{x}{2}\right)$ in terms of t.

$\sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{t^2+1}}$ $\qquad$ $\sin = \frac{\text{opposite}}{\text{hypotenuse}}$

$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{t^2+1}}$ $\qquad$ $\cos = \frac{\text{adjacent}}{\text{hypotenuse}}$

$\tan\left(\frac{x}{2}\right) = \frac{t}{1} = t$ $\qquad$ $\tan = \frac{\text{opposite}}{\text{adjacent}}$ / $\frac{\sin}{\cos}$

5) Now use double-angle formulae and write in terms of t:

$$\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$

$$\cos(x) = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$$

$$\sin(x) = 2\left(\frac{t}{\sqrt{t^2+1}}\right)\left(\frac{1}{\sqrt{t^2+1}}\right)$$

$$\sin(x) = \frac{2t}{t^2+1}$$

$$\cos(x) = \left(\frac{1}{\sqrt{t^2+1}}\right)^2 - \left(\frac{t}{\sqrt{t^2+1}}\right)^2$$

$$\cos(x) = \frac{1-t^2}{1+t^2}$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{\frac{2t}{\cancel{1+t^2}}}{\frac{1-t^2}{\cancel{1+t^2}}} = \frac{2t}{1-t^2}$$

$$\tan(x) = \frac{2t}{1-t^2}$$

$$\sin(x) = \frac{2t}{1+t^2}$$

$$\cos(x) = \frac{1-t^2}{1+t^2}$$

$$\tan(x) = \frac{2t}{1-t^2}$$

Q wants in terms of dt.

$$t = \tan\left(\frac{x}{2}\right)$$

$$\arctan(t) = \frac{x}{2}$$

$$x = 2\arctan(t)$$

$\downarrow$ use F.B. standard results

$$\frac{dx}{dt} = \frac{1}{1+t^2} \times 2$$

$$dx = \frac{2}{1+t^2}\,dt$$

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Question 5 continued

$$\int \frac{1}{2\left(\frac{2t}{1+t^2}\right)-\left(\frac{1-t^2}{1+t^2}\right)+5} \times \frac{2}{1+t^2}\, dt$$

$$\int \frac{1}{\frac{2(2t)-(1-t^2)+5(1+t^2)}{\cancel{1+t^2}}} \times \frac{2}{\cancel{1+t^2}}\, dt$$

$$\int \frac{2}{4t-1+t^2+5+5t^2}\, dt$$

$$\int \frac{2}{6t^2+4t+4}\, dt$$

$$\int \frac{1}{3t^2+2t+2}\, dt \quad // \text{ (shown)}$$

b. $\int \frac{1}{3t^2+2t+2}\, dt$ take out 3 from t^2 coefficient

When integrating quadratic @ bottom always make t^2 coefficient 1.

$$\int \frac{1}{3\left(t^2+\frac{2}{3}t+\frac{2}{3}\right)}\, dt$$

$$\frac{1}{3}\int \frac{1}{t^2+\frac{2}{3}t+\frac{2}{3}}\, dt$$

$$\frac{1}{3}\int \frac{1}{\left(t+\frac{1}{3}\right)^2-\frac{1}{9}+\frac{2}{3}}\, dt$$

$$\frac{1}{3}\int \frac{1}{\left(t+\frac{1}{3}\right)^2+\frac{5}{9}}\, dt$$

Question 5 continued

Use formulae Booklet To Compare Similar Integral

$f(x)$	$\int f(x)\,dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$

$$\frac{1}{(t+\frac{1}{3})^2+(\frac{\sqrt{5}}{3})^2} \equiv \frac{1}{a^2+x^2}$$

$x : t + \frac{1}{3}$

$a : \frac{\sqrt{5}}{3}$

sub back into integral

$$\frac{1}{\cancel{3}}\left[\frac{1}{\left(\frac{\sqrt{5}}{\cancel{3}}\right)}\arctan\left(\frac{(t+\frac{1}{3})\times 3}{(\frac{\sqrt{5}}{3})\times 3}\right)\right]$$

$$\frac{1}{\sqrt{5}}\arctan\left(\frac{3t+1}{\sqrt{5}}\right)+c$$

Sub $t = \tan\left(\frac{x}{2}\right)$ back in.

$$\frac{1}{5}\arctan\left(\frac{3\tan\left(\frac{x}{2}\right)+1}{\sqrt{5}}\right)+c$$

6·1: Taylor Series
6·2: finding Limits

6. $$y = \ln(e^{2x}\cos 3x) \qquad -\frac{1}{2} < x < \frac{1}{2}$$

(a) Show that

$$\frac{dy}{dx} = 2 - 3\tan 3x$$

(2)

(b) Determine $\dfrac{d^4y}{dx^4}$

(3)

(c) Hence determine the first 3 non-zero terms in ascending powers of x of the Maclaurin series expansion of $\ln(e^{2x}\cos 3x)$, giving each coefficient in simplest form.

(3)

(d) Use the Maclaurin series expansion for $\ln(1 + x)$ to write down the first 4 non-zero terms in ascending powers of x of the Maclaurin series expansion of $\ln(1 + kx)$, where k is a constant.

(1)

(e) Hence determine the value of k for which

$$\lim_{x\to 0}\left(\frac{1}{x^2}\ln\frac{e^{2x}\cos 3x}{1 + kx}\right)$$

exists.

(3)

a. if $y = \ln(f(x))$

$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

$\frac{d}{dx}\left(\ln\left(e^{2x}\cos 3x\right)\right) = \frac{2e^{2x}\cos(3x) - 3e^{2x}\sin(3x)}{e^{2x}\cos(3x)}$

differentiate using product rule.

$\frac{d}{dx}(\alpha\beta) = \alpha'\beta + \alpha\beta'$

where $\alpha = e^{2x}$ and $\beta = \cos(3x)$

$\frac{dy}{dx} = \frac{2\cos(3x) - 3\sin(3x)}{\cos(3x)}$

$= \cancel{\frac{2\cos(3x)}{\cos(3x)}} - \frac{3\sin(3x)}{\cos(3x)}$

$\frac{dy}{dx} = 2 - 3\tan(3x)$ // (shown)

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Question 6 continued

b. $\frac{dy}{dx} = 2 - 3\tan(3x)$

in F.B.

$$\frac{d}{dx}\left(\tan(kx)\right) = k\sec^2(kx)$$

$$\frac{d}{dx}\left(\sec(kx)\right) = k\sec(kx)\tan(kx)$$

$\frac{d^2y}{dx^2} = -9\sec^2(3x)$

$= -9\left(\sec(3x)\right)^2$

use chain rule

$\frac{d^3y}{dx^3} = -18\sec(3x) \times 3\sec(3x)\tan(3x)$

$= -54\sec^2(3x)\tan(3x)$

$= -54\left(\sec(3x)\right)^2\tan(3x)$

$\frac{d^4y}{dx^4} = -108\sec(3x)\tan(3x)3\sec(3x)\tan(3x) - 162\left(\sec(3x)\right)^2\sec^2(3x)$

$\frac{d^4y}{dx^4} = -324\sec^2(3x)\tan^2(3x) - 162\sec^4(3x)$

c. Maclaurin's series centred around 0.

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^r}{r!}f^{(r)}(0) + \ldots$ (Given in F.B.)

$y_0 = \ln\left(e^{2(0)}\cos(3(0))\right) = 0$

$\left.\frac{dy}{dx}\right|_{x=0} = 2 - 3\tan(3(0)) = 2$

$\left.\frac{d^2y}{dx^2}\right|_{x=0} = -9\sec^2(3(0)) = -9$

$\left.\frac{d^3y}{dx^3}\right|_{x=0} = -54\sec^2(3(0))\tan(3(0)) = 0$

$\left.\frac{d^4y}{dx^4}\right|_{x=0} = -324\sec^2(3(0))\tan^2(3(0)) - 162\sec^4(3(0)) = -162$

Question 6 continued

$f(x) = 0 + 2x - \frac{9x^2}{2} - \frac{162x^4}{24} + \ldots$

$f(x) = 2x - \frac{9}{2}x^2 - \frac{162}{24}x^4 + \ldots$

a. Maclaurin Series for $\ln(1+x)$ given in F.B.
↳ sub (kx) instead of (x)

$\ln(1+kx) = (kx) - \frac{(kx)^2}{2} + \frac{(kx)^3}{3} - \frac{(kx)^4}{4} + \ldots$

$\ln(1+kx) = kx - \frac{k^2x^2}{2} + \frac{k^3x^3}{3} - \frac{k^4x^4}{4} + \ldots$

e. rewrite: $\ln\left(\frac{e^{2x}\cos(3x)}{1+kx}\right) = \ln\left(e^{2x}\cos(3x)\right) - \ln(1+kx)$

worked out Maclaurin in (c)

worked out Maclaurin in (a)

$\left(2x - \frac{9}{2}x^2 - \frac{27}{4}x^4\right) - \left(kx - \frac{k^2x^2}{2} + \frac{k^3x^3}{3} - \frac{k^4x^4}{4}\right)$

$= (2-k)x - \frac{(9-k^2)x^2}{2} - \frac{(k^3)x^3}{3} - \frac{(27-k^4)x^4}{4}$

$\lim_{x\to 0}\left(\frac{1}{x^2}\left((2-k)x - \frac{(9-k^2)x^2}{2} - \frac{(k^3)x^3}{3} - \frac{(27-k^4)x^4}{4}\right)\right)$

$\lim_{x\to 0}\left(\frac{(2-k)}{x} - \frac{(9-k^2)}{2} - \frac{(k^3)x}{3} - \frac{(27-k^4)x^2}{4}\right)$

For limit to exist $2-k=0$, other terms will go to 0.

Question 6 continued

$2 - k = 0$

$k = 2$

$k = 2$

(Total for Question 6 is 12 marks)

7. With respect to a fixed origin O the point A has coordinates (3, 6, 5) and the line l has equation

$$\big(\mathbf{r} - (12\mathbf{i} + 30\mathbf{j} + 39\mathbf{k})\big) \times (7\mathbf{i} + 13\mathbf{j} + 24\mathbf{k}) = \mathbf{0}$$

The points B and C lie on l such that $AB = AC = 15$

Given that A does not lie on l and that the x coordinate of B is negative,

(a) determine the coordinates of B and the coordinates of C **(4)**

(b) Hence determine a Cartesian equation of the plane containing the points A, B and C **(3)**

The point D has coordinates $(-2, 1, \alpha)$, where α is a constant.

Given that the volume of the tetrahedron $ABCD$ is 147

(c) determine the possible values of α **(4)**

Given that $\alpha > 0$

(d) determine the shortest distance between the line l and the line passing through the points A and D, giving your answer to 2 significant figures. **(4)**

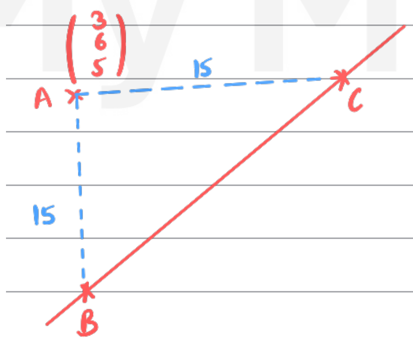
a. in form: $(r-a)\times b = 0$

↳ convert into form: $r = a + \lambda b$

$\big(r - (12\underline{i} + 30\underline{j} + 39\underline{k})\big) \times (7\underline{i} + 13\underline{j} + 24\underline{k}) = 0$

$\Rightarrow \quad r = \begin{pmatrix} 12 \\ 30 \\ 39 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 13 \\ 24 \end{pmatrix}$

$r = \begin{pmatrix} 12 \\ 30 \\ 39 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 13 \\ 24 \end{pmatrix}$



Call B and C general points on line: $r = \begin{pmatrix} 12 + 7\lambda \\ 30 + 13\lambda \\ 39 + 24\lambda \end{pmatrix}$

$\overrightarrow{OB}$ or $\overrightarrow{OC}$: $r = \begin{pmatrix} 12 + 7\lambda \\ 30 + 13\lambda \\ 39 + 24\lambda \end{pmatrix}$

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Question 7 continued

$\vec{AB} = \vec{AO} + \vec{OB}$

$= \begin{pmatrix} -3 \\ -6 \\ -5 \end{pmatrix} + \begin{pmatrix} 12+7k \\ 30+13k \\ 39+24k \end{pmatrix}$

$\vec{AB} = \begin{pmatrix} 9+7k \\ 24+13k \\ 34+24k \end{pmatrix}$

$|\vec{AB}| = 15$

$\therefore \sqrt{(9+7k)^2+(24+13k)^2+(34+24k)^2} = 15$

$(9+7k)^2+(24+13k)^2+(34+24k)^2 = 225$

$49k^2+126k+81+169k^2+624k+576+576k^2+1632k+1156 = 225$

$794k^2+2382k+1588 = 0$

(Solve for k w/ quadratic formulae.

$k = \dfrac{-(2382) \pm \sqrt{(2382)^2-(4)(794)(1588)}}{2(794)}$

$k = -1$ or $k = -2$

sub back into gen eqn line to find coord. B and C.

$\begin{pmatrix} 12+7(-1) \\ 30+13(-1) \\ 39+24(-1) \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 15 \end{pmatrix}$

$\begin{pmatrix} 12+7(-2) \\ 30+13(-2) \\ 39+24(-2) \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ -9 \end{pmatrix}$

Question 7 continued

Q states B has -v.e. x-coord.

$\therefore$ B: $\begin{pmatrix}-2\\4\\-9\end{pmatrix}$ and C: $\begin{pmatrix}5\\17\\15\end{pmatrix}$

b. In form $ax + by + cz = d$

$r \cdot n = d$

- n is normal vector perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
  - cross product $\overrightarrow{AB}$ and $\overrightarrow{AC}$ to work out normal vector, n.
- d is a constant, worked out by using scalar product of any point on plane (A, B, C) and normal vector, n.

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= \begin{pmatrix}-3\\-6\\-5\end{pmatrix} + \begin{pmatrix}5\\17\\15\end{pmatrix}$

$= \begin{pmatrix}2\\11\\10\end{pmatrix}$

$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$= \begin{pmatrix}-3\\-6\\-5\end{pmatrix} + \begin{pmatrix}-2\\4\\-9\end{pmatrix}$

$= \begin{pmatrix}-5\\-2\\-14\end{pmatrix}$

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix}2\\11\\10\end{pmatrix} \times \begin{pmatrix}-5\\-2\\-14\end{pmatrix}$

$= \begin{vmatrix} i & j & k \\ 2 & 11 & 10 \\ -5 & -2 & -14 \end{vmatrix} = [(11)(-14) - (10)(-2)]\underline{i} + [(10)(-5) - (2)(-14)]\underline{j} + [(2)(-2) - (-5)(11)]\underline{k}$

$= -134\underline{i} - 22\underline{j} + 51\underline{k}$ ← normal vector n

$\begin{pmatrix}3\\6\\5\end{pmatrix} \cdot \begin{pmatrix}-134\\-22\\51\end{pmatrix} = (3)(-134) + (6)(-22) + (5)(51) = -279$

$-134x - 22y + 51z = -279$

DO NOT WRITE IN THIS AREA

Question 7 continued

c. volume of tetrahedron: $\frac{1}{6}\left|\vec{AD}\cdot\underbrace{(\vec{AB}\times\vec{AC})}_{\text{calculated in (b.)}}\right|$

← (must memorise formulaes for volume of tetrahedron, parallelepiped and area of triangle and parallelogram.)

$\vec{AD} = \vec{AO} + \vec{OD}$

$= \begin{pmatrix}-3\\-6\\-5\end{pmatrix} + \begin{pmatrix}-2\\1\\\alpha\end{pmatrix}$

$= \begin{pmatrix}-5\\-5\\\alpha-5\end{pmatrix}$

$\frac{1}{6}\left|\begin{pmatrix}-5\\-5\\\alpha-5\end{pmatrix}\cdot\begin{pmatrix}-134\\-22\\51\end{pmatrix}\right| = 147$

$|(670)+(110)+51\alpha-255| = 882$

$|51\alpha+525| = 882$

$51\alpha+525 = 882$

$51\alpha = 357$

$\alpha = 7$

$-(51\alpha+525) = 882$

$-51\alpha-525 = 882$

$-51\alpha = 1407$

$\alpha = -\frac{469}{17}$

$\alpha = 7$ or $\alpha = -469/17$

a. if $\alpha > 0$, $\alpha = 7$

$\therefore D = \begin{pmatrix}-2\\1\\7\end{pmatrix}$

use formulae: $\left|\frac{(a-c)\cdot(b\times d)}{|b\times d|}\right|$ for $r = a+\lambda b$, $r = c+\mu d$

This is the formulae to work out distance between 2 skew lines.

- Not in F.B. (Memorise)

Question 7 continued

$r: \begin{pmatrix}12\\30\\39\end{pmatrix} + \lambda\begin{pmatrix}7\\13\\24\end{pmatrix}$ ← line l

line AD:

$r: \begin{pmatrix}3\\6\\5\end{pmatrix} + \mu\begin{pmatrix}-5\\-5\\2\end{pmatrix}$

direction vector: $\vec{AD} = \vec{AO} + \vec{OD}$

$= \begin{pmatrix}-3\\-6\\-5\end{pmatrix} + \begin{pmatrix}-2\\1\\7\end{pmatrix}$

$= \begin{pmatrix}-5\\-5\\2\end{pmatrix}$

$$\left| \frac{\begin{pmatrix}-9\\-24\\-34\end{pmatrix} \cdot \left(\begin{pmatrix}-5\\-5\\2\end{pmatrix} \times \begin{pmatrix}7\\13\\24\end{pmatrix}\right)}{\left|\begin{pmatrix}-5\\-5\\2\end{pmatrix} \times \begin{pmatrix}7\\13\\24\end{pmatrix}\right|} \right|$$

$$\begin{pmatrix}-5\\-5\\2\end{pmatrix} \times \begin{pmatrix}7\\13\\24\end{pmatrix}$$

$$= \begin{vmatrix} i & j & k \\ -5 & -5 & 2 \\ 7 & 13 & 24 \end{vmatrix} = [(-5)(24) - (2)(13)]\underline{i} + [(2)(7) - (-5)(24)]\underline{j} + [(-5)(13) - (-5)(7)]\underline{k}$$

$= -146\underline{i} + 134\underline{j} - 30\underline{k}$

must use dot product now. $\begin{pmatrix}a\\b\\c\end{pmatrix} \cdot \begin{pmatrix}d\\e\\f\end{pmatrix} = ad + be + cf$

$$\left| \frac{\begin{pmatrix}-9\\-24\\-34\end{pmatrix} \cdot \begin{pmatrix}-146\\134\\-30\end{pmatrix}}{\sqrt{(-146)^2 + (134)^2 + (-30)^2}} \right| = \left| \frac{1314 - 3216 + 1020}{22\sqrt{83}} \right| = \left| \frac{-882}{22\sqrt{83}} \right|$$

$$= \frac{882}{22\sqrt{83}}$$

cannot have -v.e. distance so must change sign to +v.e.

≈ 4.4 (2 s.f.)

(Total for Question 7 is 15 marks)

TOTAL FOR PAPER IS 75 MARKS